Algebraic Proof - Worksheet

Skill

Group A - Proving identities

Prove that:

1) $2(4a + 2b) + 3(2a - 3b) \equiv 14a - 5b$

2) $3(2a + 4b) + 3(3a + 5b) \equiv 15a + 27b$

3) $4(3a + 2b) - 2(2a + 2b) \equiv 8a + 4b$

4) $(n + 3)^2 \equiv n^2 + 6n + 9$

5) $(5n + 3)(n - 7) \equiv 5n^2 - 32n - 21$

6) $(n + 4)^2 - (3n + 4) \equiv (n + 1)(n + 4) + 8$

7) $(n + 4)^2 - (3n + 4) \equiv (n + 2)(n + 3) + 6$

8) $(n + 3)^2 - (3n + 5) \equiv (n + 1)(n + 2) + 2$

9) $(n - 5)^2 - (2n - 1) \equiv (n - 3)(n - 9) - 1$

10) $(n - 3)^2 - (n - 5) \equiv (n - 3)(n - 4) + 2$

11) $(2n - 1)^2 + (2n + 1)^2 \equiv 8n^2 + 2$

12) $(3n + 2)^2 - (n + 2)^2 \equiv 8n(n + 1)$

Group B - Proving properties of number

Prove algebraically that:

1) The sum of an odd number and an even number is odd.

2) The product of an odd number and an even number is even.

3) The product of any two odd numbers is odd.

4) The product of any two even numbers is even.

5) The sum of an odd number and an odd number is an even number.

6) The sum of four consecutive whole numbers is always even.

7) $(n + 4)^2 - (n + 2)^2$ is even for all positive integer values of n.

8) $(2n + 1)^2 - (2n + 1)$ is even for all positive integer values of n.

9) $(2n - 1)(3n - 2) - (6n - 1)(n - 2)$ is always even.

10) $(4n + 1)^2 - (2n - 1)$ is an even number for all positive values of n.

11) $3n(3n + 4) + (n - 6)^2$ is positive for all values of n.

12) $(2n + 3)^3 - (2n + 1)$ is always even for all positive values of n.

Algebraic Proof - Worksheet

Group C - Proving multiples

Prove the following, for all positive integers values of n:

1) $15n$ is always a multiple of 5

2) $42n$ is always a multiple of 6

3) $33n$ is always a multiple of 3

4) $12n + 18$ is always a multiple of 6

5) $35n + 28$ is always a multiple of 7

6) $9n^2 + 36n + 81$ is always a multiple of 9

7) $(5n - 1)^2 + 3(3 - 10n)$ is always a multiple of 5

8) $(4n + 2)^2 - 12(n + 1)$ is always a multiple of 4

9) $(7n + 4)^2 - (7n - 4)^2$ is always a multiple of 8

10) $(8n + 2)^2 - (8n - 3)^2$ is always a multiple of 5

11) $(6n + 5)^2 - (6n - 2)^2$ is always a multiple of 21

12) $(3n + 1)^2 - (3n - 1)^2$ is always a multiple of 4

Algebraic Proof - Worksheet

Applied

1) Prove that the square of any odd number is always one more than a multiple of 4.

2) Show that the sum of any three consecutive multiples of 3 is also a multiple of 3.

3) Prove algebraically that the sum of the square of two consecutive odd numbers is even.

4) **(a)** Expand and simplify: $(a - b)(a + b - 1)$

(b) Show that $(2a - 1)^2 - (2b - 1)^2 = 4(a - b)(a + b - 1)$

5) **(a)** A Fibonacci sequence is formed by adding the previous two terms to get the next term. Continue the Fibonacci sequence $1, 1, 2, 3,$ ... up to ten terms.

(b) Continue the Fibonacci sequence, $a, b, a + b, a + 2b,$ ... up to ten terms.

(c) Prove that the difference between the 8th and the 5th term of any Fibonacci sequence is twice the 6^{th} term.

Algebraic Proof - Exam Questions

1) Prove that $(2n + 3)^2 - (2n - 3)^2$

is a multiple of 8 for all positive integer values of n.

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(3 marks)

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2) Prove algebraically that the difference between the squares of any two consecutive integers is equal to the sum of these two integers.

…………………

(5 marks)

Algebraic Proof - Exam Questions

3) Prove that $(n + 1)^2 - (n - 1)^2 + 1$

is always odd for all positive integer values of n.

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(3 marks)

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4) The product of two consecutive positive integers is added to the larger of the two integers.

Prove that the result is always a square number.

....

(3 marks)

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Algebraic Proof - Answers

	Question	Answer
	Skill Questions	
Group A	Prove that: **1)** $2(4a + 2b) + 3(2a - 3b)$ $\equiv 14a - 5b$	**1)** $2(4a + 2b) + 3(2a - 3b)$ $= 8a + 4b + 6a - 9b$ $= 14a - 5b$ $\therefore 2(4a + 2b) + 3(2a - 3b)$ $\equiv 14a - 5b$
	2) $3(2a + 4b) + 3(3a + 5b)$ $\equiv 15a + 27b$	**2)** $3(2a + 4b) + (3a + 5b)$ $= 6a + 12b + 9a + 15b$ $15a + 27b$ $\therefore 3(2a + 4b) + 3(3a + 5b)$ $\equiv 15a + 27b$
	3) $4(3a + 2b) - 2(2a + 2b)$ $\equiv 8a + 4b$	**3)** $4(3a + 2b) - 2(2a + 2b)$ $= 12a + 8b - 4a - 4b$ $= 8a + 4b$ $\therefore 4(3a + 2b) - 2(2a + 2b)$ $\equiv 8a + 4b$
	4) $(n + 3)^2 \equiv n^2 + 6n + 9$	**4)** $(n + 3)^2$ $= (n + 3)(n + 3)$ $= n^2 + 3n + 3n + 9$ $= n^2 + 6n + 9$ $\therefore (n + 3)^2 \equiv n^2 + 6n + 9$
	5) $(5n + 3)(n - 7)$ $\equiv 5n^2 - 32n - 21$	**5)** $(5n + 3)(n - 7)$ $= 5n^2 - 35n + 3n - 21$ $= 5n^2 - 32n - 21$ $\therefore (5n + 3)(n - 7)$ $\equiv 5n^2 - 32n - 21$

Algebraic Proof - Answers

Group A contd	**6)** $(n + 4)^2 - (3n + 4)$ $\equiv (n + 1)(n + 4) + 8$	**6)** $(n + 4)^2 - (3n + 4)$ $= (n + 4)(n + 4) - (3n + 4)$ $= n^2 + 8n + 16 - 3n - 4$ $= n^2 + 5n + 12$ $= n^2 + 5n + 4 + 8$ $= (n + 1)(n + 4) + 8$ $\therefore (n + 4)^2 - (3n + 4)$ $\equiv (n + 1)(n + 4) + 8$
	7) $(n + 4)^2 - (3n + 4)$ $\equiv (n + 2)(n + 3) + 6$	**7)** $(n + 4)^2 - (3n + 4)$ $= (n + 4)(n + 4) - (3n + 4)$ $= n^2 + 8n + 16 - 3n - 4$ $= n^2 + 5n + 12$ $= n^2 + 5n + 6 + 6$ $= (n + 2)(n + 3) + 6$ $\therefore (n + 4)^2 - (3n + 4)$ $\equiv (n + 2)(n + 3) + 6$
	8) $(n + 3)^2 - (3n + 5)$ $\equiv (n + 1)(n + 2) + 2$	**8)** $(n + 3)^2 - (3n -+ 5)$ $= (n + 3)(n + 3) - (3n + 5)$ $= n^2 + 6n + 9 - 3n - 5$ $= n^2 + 3n + 4$ $= n^2 + 3 + 2 + 2$ $\therefore (n + 3)^2 - (3n + 5)$ $\equiv (n + 1)(n + 2) + 2$

Algebraic Proof - Answers

Group A contd	**9)** $(n-5)^2-(2n-1) \equiv (n-3)(n-9)-1$	**9)** $(n-5)^2-(2n-1)$ $=(n-5)(n-5)-(2n-1)$ $=n^2-10n+25-2n+1$ $=n^2-12n+26$ $=n^2-12n+27-1$ $=(n-3)(n-9)-1$ $\therefore (n-5)^2-(2n-1)$ $\equiv (n-3)(n-9)-1$
	10) $(n-3)^2-(n-5)$ $\equiv (n-3)(n-4)+2$	**10)** $(n-3)^2-(n-5)$ $=(n-3)(n-3)-(n-5)$ $=n^2-6n+9-n+5$ $=n^2-7n+14$ $=n^2-7n+12+2$ $=(n-3)(n-4)+2$ $\therefore (n-3)^2-(n-5)$ $\equiv (n-3)(n-4)+2$
	11) $(2n-1)^2+(2n+1)^2$ $\equiv 8n^2+2$	**11)** $(2n-1)^2+(2n+1)^2$ $=(2n-1)(2n-1)+(2n+1)(2n+1)$ $=4n^2-4n+1+4n^2+4n+1$ $=8n^2+2$ $\therefore (2n-1)^2+(2n+1)^2$ $\equiv 8n^2+2$
	12) $(3n+2)^2-(n+2)^2$ $\equiv 8n(n+1)$	**12)** $(3n+2)^2-(n+2)^2$ $=(3n+2)(3n+2)-(n+2)(n+2)$ $=9n^2+12n+4-(n^2+4n+4)$ $=9n^2+12n+4-n^2-4n-4$ $=8n^2+8n$ $=8n(n+1)$ $\therefore (3n+2)^2-(n+2)^2$ $\equiv 8n(n+1)$

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Algebraic Proof - Answers

Group B	Prove algebraically that:	
	1) The sum of an odd number and an even number is odd.	**1)** $2n + 1$ = odd, $2m$ = even $2n + 1 + 2m = 2(n + m) + 1$ Multiple of 2, +1, so the sum of an odd number and an even number is odd.
	2) The product of an odd number and an even number is even.	**2)** $2n + 1$ = odd, $2m$ = even $2m(2n + 1) = 4mn + 2m$ $= 2(2mn + m)$ Multiple of 2, so the product of an odd number and an even number is even.
	3) The product of any two odd numbers is odd.	**3)** $2n + 1$ = odd, $2m + 1$ = odd $(2n + 1)(2m + 1)$ $= 4mn + 2n + 2m + 1$ $= 2(2mn + n + m) + 1$ Multiple of 2, + 1, so the product of any two odd numbers is odd.
	4) The product of any two even numbers is even.	**4)** $2n$ = even, $2m$ = even $2n \times 2m = 4nm$ $= 2(2mn)$ Multiple of 2, so the product of any two even numbers is even.
	5) The sum of an odd number and an odd number is an even number.	**5)** $2n + 1$ = odd, $2m + 1$ = odd $(2n + 1) + (2m + 1)$ $= 2n + 2m + 2$ $= 2(n + m + 1)$ Multiple of 2, so the sum of an odd number and an odd number is an even number.
	6) The sum of four consecutive whole numbers is always even.	**6)** $(n) + (n + 1) + (n + 2) + (n + 3)$ $= 4n + 6$ $= 2(2n + 3)$ $2(2n + 3)$ is a multiple of 2 ∴ the sum of four consecutive whole numbers is even

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Algebraic Proof - Answers

Group B contd	**7)** $(n+4)^2 - (n+2)^2$ is even for all positive integer values of n.	**7)** $(n+4)^2 - (n+2)^2$ $= (n+4)(n+4) - (n+2)(n+2)$ $= n^2 + 8n + 16 - n^2 - 4n - 4$ $= 4n + 12$ $= 2(2n+6)$ $2(2n+6)$ is a multiple of 2 $\therefore (n+4)^2 - (n+2)^2$ is even for all positive integer values of n.
	8) $(2n+1)^2 - (2n+1)$ is even for all positive integer values of n.	**8)** $(2n+1)^2 - (2n+1)$ $= (2n+1)(2n+1) - (2n+1)$ $= 4n^2 + 4n + 1 - 2n - 1$ $= 4n^2 + 2n$ $= 2(2n^2 + n)$ $2(2n^2+n)$ is a multiple of 2 $\therefore (2n+1)^2 - (2n+1)$ is even for all positive integer values of n.
	9) $(2n-1)(3n-2) - (6n-1)(n-2)$ is always even.	**9)** $(2n-1)(3n-2) - (6n-1)(n-2)$ $= 6n^2 - 4n - 3n + 2 - 6n^2 + 13n - 2$ $= 6n$ $= 2(3n)$ $2(3n)$ is a multiple of 2 $\therefore (2n-1)(3n-2) - (6n-1)(n-2)$ is always even
	10) $(4n+1)^2 - (2n-1)$ is an even number for all positive values of n.	**10)** $(4n+1)^2 - (2n-1)$ $= (4n+1)(4n+1) - (2n-1)$ $= 16n^2 + 8n + 1 - 2n + 1$ $= 16n^2 + 6n + 2$ $= 2(8n^2 + 3n + 1)$ $2(8n^2+3n+1)$ is a multiple of 2 $\therefore (4n+1)^2 - (2n-1)$ is an even number for all positive values of n.

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Algebraic Proof - Answers

Group B contd	**11)** $3n(3n + 4) + (n - 6)^2$ is positive for all values of n.	**11)** $3n(3n + 4) + (n - 6)^2$ $= 9n^2 + 12n + (n - 6)(n - 6)$ $= 9n^2 + 12n + n^2 - 12n + 36$ $= 10n^2 + 36$ $n^2 \geq 0$ always so $10n^2 \geq 0$ and so $10n^2 + 16 > 0$ $\therefore 3n(3n + 4) + (n - 6)^2$ is always positive
	12) $(2n + 3)^3 - (2n + 1)$ is always even for all positive values of n.	**12)** $(2n + 3) - (2n + 1)$ $= (2n + 3)(2n + 3)(2n + 3) - (2n + 1)$ $= (4n^2 + 12n + 9)(2n + 3) - 2n - 1$ $= 8n^3 + 36n^2 + 54n + 27 - 2n - 1$ $= 8n^3 + 36n^2 + 52n + 26$ $= 2(4n^3 + 18n + 26n + 13)$ $2(4n^3 + 18n + 26n + 13)$ is a multiple of 2 $\therefore (2n + 3)^3 - (2n + 1)$ is always even for all positive values of n
Group C	Prove the following, for all positive integers values of n:	
	1) $15n$ is always a multiple of 5	**1)** $15n = 5 \times 3n$ $\therefore 15n$ is always a multiple of 5
	2) $42n$ is always a multiple of 6	**2)** $42n = 6 \times 7n$ $\therefore 42n$ is always a multiple of 6
	3) $33n$ is always a multiple of 3	**3)** $33n = 3 \times 11n$ $\therefore 33n$ is always a multiple of 3
	4) $12n + 18$ is always a multiple of 6	**4)** $12n + 18 = 6 \times (2n + 3)$ $\therefore 12n + 18$ is always a multiple of 6
	5) $35n + 28$ is always a multiple of 7	**5)** $35n + 28 = 7 \times (5n + 4)$ $\therefore 35n + 28$ is always a multiple of 7
	6) $9n^2 + 36n + 81$ is always a multiple of 9	**6)** $9n^2 + 36n + 81 = 9 \times (n^2 + 4n + 9)$ $\therefore 9n^2 + 36n + 81$ is always a multiple of 9

Algebraic Proof - Answers

Group C	**7)** $(5n - 1)^2 + 3(3 - 10n)$ is always a multiple of 5	**7)** $(5n-1)^2+3(3-10n)$ $=(5n-1)(5n-1)+9-30n$ $=25n^2-10n+1+9-30n$ $=25n^2-40n+10$ $=5(5n^2-8n+2)$ ∴Multiple of 5
	8) $(4n + 2)^2 - 12(n + 1)$ is always a multiple of 4	**8)** $(4n+2)^2-12(n+1)$ $=(4n+2)(4n+2)-12n-12$ $=16n^2+16n+4-12n-12$ $=16n^2+4n-8$ $=4(4n^2+n-2)$ ∴Multiple of 4
	9) $(7n + 4)^2 - (7n - 4)^2$ is always a multiple of 8	**9)** $(7n+4)^2-(7n-4)^2$ $=(7n+4)(7n+4)-(7n-4)(7n-4)$ $=49n^2+56n+16-(49n^2-56n+16)$ $=49n^2+56n+16-49n^2+56n-16$ $=112n$ $=8(14n)$ ∴Multiple of 8
	10) $(8n + 2)^2 - (8n - 3)^2$ is always a multiple of 5	**10)** $(8n+2)^2-(8n-3)^2$ $=(8n+2)(8n+2)-(8n-3)(8n-3)$ $=64n^2+32n+4-(64n^2-48n+9)$ $=64n^2+32n+4-64n^2+48n-9$ $=80n-5$ $=5(16n-1)$ ∴Multiple of 5

Algebraic Proof - Answers

Group C contd	**11)** $(6n+5)^2-(6n-2)^2$ is always a multiple of 21	**11)** $(6n+5)^2-(6n-2)^2$ $=(6n+5)(6n+5)-(6n-2)(6n-2)$ $=36n^2+60n+25-(36n^2-24n+4)$ $=36n^2+60n+25-36n^2+24n-4$ $=84n+21$ $=21(4n+1)$ ∴Multiple of 21
	12) $(3n+1)^2-(3n-1)^2$ is always a multiple of 4	**12)** $(3n+1)^2-(3n-1)^2$ $=(3n+1)(3n+1)-(3n-1)(3n-1)$ $=9n^2+6n+1-(9n^2-6n+1)$ $=9n^2+6n+1-9n^2+6n-1$ $=12n$ $=4(3n)$ ∴Multiple of 4

Algebraic Proof - Answers

	Question	Answer
	Applied Questions	
1)	Prove that the square of any odd number is always one more than a multiple of 4.	$2n + 1$ **is odd** $(2n + 1)^2$ $= (2n + 1)(2n + 1)$ $= 4n^2 + 4n + 1$ $= 4(n^2 + n) + 1$ $\therefore$ 1 more than a multiple of 4
2)	Show that the sum of any three consecutive multiples of 3 is also a multiple of 3.	$3(n) = 3n$ $3(n + 1) = 3n + 3$ $3(n + 2) = 3n + 6$ $3n + 3n + 3 + 3n + 6$ $= 9n + 9$ $= 3(n + 3)$ $\therefore$ a multiple of 3
3)	Prove algebraically that the sum of the square of two consecutive odd numbers is even.	$(2n + 1)^2 + (2n + 3)^2$ $4n^2 + 4n + 1 + 4n^2 + 12n + 9$ $8n^2 + 16n + 10$ $2(4n^2 + 8n + 5)$ $\therefore (2n + 1)^2 + (2n + 3)^2$ is always even

Algebraic Proof - Answers

4)	**a)** Expand and simplify: $(a - b)(a + b - 1)$ **b)** Show that $(2a - 1)^2 - (2b - 1)^2 = 4(a - b)(a + b - 1)$	**a)** $a^2 - a - b^2 + b$ **b)** $(2a - 1)^2 - (2b - 1)^2$ $= 4a^2 - 4a + 1 - (4b^2 - 4b + 1)$ $= 4a^2 - 4a + 1 - 4b^2 + 4b - 1$ $= 4a^2 - 4a - 4b^2 + 4b$ $= 4(a^2 - a - b^2 + b)$ $= 4(a - b)(a + b - 1)$
5)	**a)** A Fibonacci sequence is formed by adding the previous two terms to get the next term. Continue the Fibonacci sequence $1, 1, 2, 3,$ ... up to ten terms. **b)** Continue the Fibonacci sequence, $a, b, a + b, a + 2b,$ ... up to ten terms. **c)** Prove that the difference between the 8^{th} and the 5^{th} term of any Fibonacci sequence is twice the 6^{th} term.	**a)** $1, 1, 2, 3, 5, 8, 13, 21, 34, 55$ **b)** $a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, 5a + 8b, 13a + 21b, 21a + 34b$ **c)** 8^{th} term: $8a + 13b$ 5^{th} term: $2a + 3b$ 6^{th} term: $3a + 5b$ Difference: $8a + 13b - (2a + 3b)$ $= 8a + 13b - 2a - 3b$ $= 6a + 10b$ $= 2(3a + 5b)$

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Algebraic Proof - Mark Scheme

	Question	Answer	
	Exam Questions		
1)	Prove that $(2n+3)^2-(2n-3)^2$ is a multiple of 8 for all positive integer values of n.	$4n^2+12n+9$ **or** $4n^2-12n+9$ **or** $-4n^2+12n-9$ **seen** $24n$ $8(3n)$	**(1)** **(1)** **(1)**
2)	Prove algebraically that the difference between the squares of any two consecutive integers is equal to the sum of these two integers.	Two consecutive integers written algebraically e.g. n **and** $n+1$, **or** $n-1$ and n, **or** $n+1$ **and** $n+2$ etc The difference between the squares of their two integers written algebraically e.g. $(n+1)^2-n^2$ **or** $n^2-(n-1)^2$ **or** $(n+2)^2-(n+1)^2$ **etc** Correct expansion e.g. $n^2+2n+1-n^2$ **or** n^2-n^2+2n-1 **or** $n^2+4n+4-n^2-2n-1$ Correct simplification e.g. $2n+1$ **or** $2n-1$ **or** $2n+3$ Correct sum of their two integers is equivalent to their simplification e.g. $(n+1)+n=2n+1$ **or** $n+n-1=2n-1$ **or** $(n+1)+(n+2)=2n+3$ **etc**	**(1)** **(1)** **(1)** **(1)** **(1)**
3)	Prove that $(n+1)^2-(n-1)^2+1$ is always odd for all positive integer values of n.	$(n+1)^2-(n-1)^2+1$ $=n^2+2n+1-(n^2-2n+1)+1$ **or** $n^2+2n+1-n^2+2n-1+1$ $=4n+1$ $4n$ is even **oe and so** $4n+1$ is odd **oe**	**(1)** **(1)** **(1)**

Algebraic Proof - Mark Scheme

4)	The product of two consecutive positive integers is added to the larger of the two integers. Prove that the result is always a square number.	For n and $n + 1$: $n(n + 1) + (n + 1)$ $= n^2 + n + n + 1$ $= n^2 + 2n + 1$ $(n + 1)^2$	 **(1)** **(1)** **(1)**